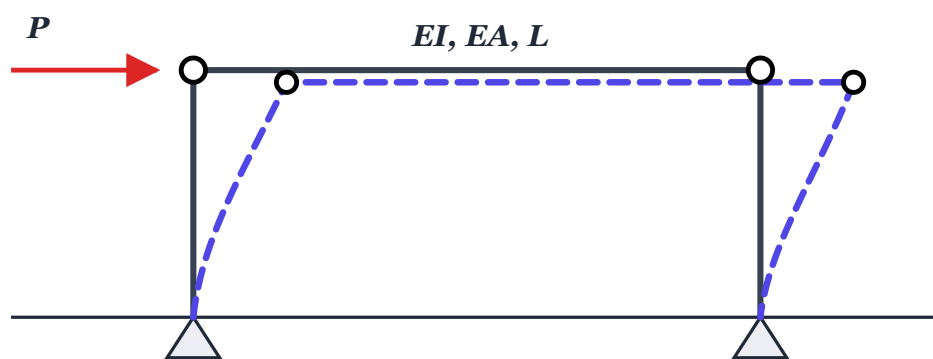




STIFFSTRUCT — THEORY NOTES

The Direct Stiffness Method

Element matrices, transformations, loads and sign conventions



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Companion document to the StiffStruct web application. Educational tool: results must be checked by a qualified engineer.

Contents

	Introduction and notation	3
1	Overview and assumptions	4
2	Coordinate systems and sign conventions	5
3	Degrees of freedom and numbering	6
4	Element stiffness matrices	7
4.1	Axial and torsional bars	7
4.2	Beam element, 4×4	7
4.3	2D truss element (global 4×4)	7
4.4	2D frame element (6×6)	8
4.5	3D truss element (global 6×6)	8
4.6	3D frame element (12×12)	8
5	Assembly	10
6	Partitioned equations, displacements and reactions	11
7	Fixed-end forces of member loads	12
7.1	Work-equivalent loads	12
7.2	Closed forms (downward loads, magnitudes P, w ; a from node i , $b = L - a$)	12
7.3	Moment loads	13
7.4	Axial loads, self-weight	13
7.5	Temperature changes and lack of fit	13
8	Member end releases (static condensation)	14
9	Springs and inclined supports	15
9.1	Elastic springs	15
9.2	Inclined supports and the nodal rotation Γ	15
9.3	Auto-restrained dofs	15
10	Prescribed displacements (settlement)	16
11	Member end forces	17
12	Exact internal-force and deflection functions	18
13	Self-checks	19
14	Load combinations and envelopes	20
15	Limitations and references	21

Introduction and notation

This document derives every formula the solver in `js/solver/` implements, in the same notation as the code (`elements.js`, `loads.js`, `analyze.js`, `postprocess.js`) and the same sign conventions as `ARCHITECTURE.md` §2. Section numbers below are referenced from comments in the code and from the step-by-step report.

Notation: bold lower case = vectors, bold upper case = matrices. A prime marks a quantity in member **local** axes ($\mathbf{k}'$, $\mathbf{u}'$, $\mathbf{q}'$); quantities without a prime are in **global** axes. E Young's modulus, $G = E/(2(1 + \nu))$ shear modulus, A area, I_z, I_y second moments of area about the local z and y axes, J torsion constant, L member length, α coefficient of thermal expansion.

1 Overview and assumptions

The program solves **linear static** problems for skeletal structures built from prismatic members joined at nodes:

ANALYSIS TYPE	ELEMENT	WHAT IT MODELS
beam	beam, 2 dofs/node	continuous beams in a plane (transverse load only)
truss2d	bar, 2 dofs/node	pin-jointed plane trusses
frame2d	frame, 3 dofs/node	plane frames (axial + bending)
truss3d	bar, 3 dofs/node	pin-jointed space trusses
frame3d	frame, 6 dofs/node	space frames (axial, 2-axis bending, torsion)

Assumptions (all of them are *limitations* too, see §15):

1. **Linear elastic** material, Hooke's law, no yielding or cracking.
2. **Small displacements**: equilibrium is written on the undeformed geometry; no $P-\Delta$ or $P-\delta$ effects, no geometric stiffness. Superposition of load cases is exact.
3. **Euler–Bernoulli beam theory**: plane sections remain plane and normal to the deformed axis; shear deformation and rotary inertia are neglected. The exact solution of $EI v'''' = q$ for a prismatic member is a cubic in v between loads, so the Hermite cubic element is *exact* for any load that is applied with the same shape functions (§7), and no mesh refinement within a member is ever needed.
4. **Prismatic** members: E , A , I_z , I_y , J are constant along each member. Warping torsion is not modelled (St. Venant torsion only).
5. Joints are rigid unless a member end is **released** (§8). Rigid-body offsets and rigid zones are not modelled.

2 Coordinate systems and sign conventions

Global axes. X to the right, Y up, Z towards the viewer (right-handed). 2D models lie in the X – Y plane; beam models lie on the X axis. Gravity acts in $-Y$. Positive rotations follow the right-hand rule (in 2D: counter-clockwise, about $+Z$).

Member local axes. Local x runs from node i to node j with unit vector $\mathbf{e}_x = (\mathbf{x}_j - \mathbf{x}_i)/L$.

- 2D: $\mathbf{e}_y$ is $\mathbf{e}_x$ rotated $+90^\circ$, $\mathbf{e}_y = (-e_{x,2}, e_{x,1}, 0)$, $\mathbf{e}_z = +Z$. A member pointing right has local $y = \text{global } Y$.
- 3D, non-vertical member ($|\mathbf{e}_x \cdot \hat{Y}| \leq 0.999$): $\mathbf{e}_y$ is the projection of $\hat{Y}$ on the plane normal to $\mathbf{e}_x$,

$$\mathbf{e}_y = \frac{\hat{Y} - (\mathbf{e}_x \cdot \hat{Y}) \mathbf{e}_x}{\|\hat{Y} - (\mathbf{e}_x \cdot \hat{Y}) \mathbf{e}_x\|}, \quad \mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y.$$

- 3D, vertical member ($|\mathbf{e}_x \cdot \hat{Y}| > 0.999$): the projection above is undefined, so $\mathbf{e}_y = -\hat{X}$ when $\mathbf{e}_x$ points up ($+Y$) and $\mathbf{e}_y = +\hat{X}$ when it points down, and again $\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$. For both a rising and a falling column this gives $\mathbf{e}_z = +\hat{Z}$.
- Roll angle β (3D only): after the rule above, $\mathbf{e}_y$ and $\mathbf{e}_z$ are rotated about $\mathbf{e}_x$,

$$\mathbf{e}'_y = \cos \beta \mathbf{e}_y + \sin \beta \mathbf{e}_z, \quad \mathbf{e}'_z = -\sin \beta \mathbf{e}_y + \cos \beta \mathbf{e}_z.$$

A roll of 90° therefore makes local y coincide with the old local z : bending in the vertical plane of a horizontal member is then resisted by I_y instead of I_z .

Section properties. I_z is the second moment about local z (bending in the local x – y plane — the "strong axis" for a normally oriented section), I_y about local y , J the torsion constant.

Member end forces (the stiffness-method convention) are the components of the force and moment acting on the member at its ends, along the member's local axes, positive in the positive local directions (moments by the right-hand rule). The end force vector of a member is denoted $\mathbf{q}'$.

Internal forces are functions of x obtained from the free body $0 \rightarrow x$:

- N axial force, **tension positive**;
- M_z bending moment about local z , *positive-face* convention (for a horizontal 2D member with local y up it is **sagging-positive**);
- V_y shear in local y defined by $V_y = dM_z/dx$ (so a simply supported beam under gravity has positive shear at its left support);
- $M_y, V_z = dM_y/dx$ and torsion T (positive-face, right-hand rule about $+x$) in 3D.

Displacement functions along a member (local components): axial d_x , transverse d_y, d_z , twist $r_x, r_z = dd_y/dx$ and $r_y = -dd_z/dx$. Curvatures are $d^2d_y/dx^2 = M_z/(EI_z) + \kappa_z$ and $d^2d_z/dx^2 = -M_y/(EI_y) + \kappa_y$, where κ are initial (thermal) curvatures (§7.5).

Thermal loads. ΔT is the uniform temperature change, $\Delta T_y = T(+y \text{ face}) - T(-y \text{ face})$ over the depth h_y , and ΔT_z likewise over the width h_z . The free thermal curvature is $\kappa_z = -\alpha \Delta T_y/h_y$ (top hotter $\Rightarrow$ hogging) and $\kappa_y = -\alpha \Delta T_z/h_z$.

Diagram drawing side (tension or compression side) is a display setting, never a data convention.

3 Degrees of freedom and numbering

Every node has up to six dofs in the canonical order $[u_x, u_y, u_z, \theta_x, \theta_y, \theta_z]$ (labels `ux uy uz rx ry rz`). An analysis type uses a subset:

TYPE	DOFS PER NODE N_D	ELEMENT DOFS
beam	u_y, θ_z	$[v_i, \theta_i, v_j, \theta_j]$
truss2d	u_x, u_y	$[u_i, v_i, u_j, v_j]$ (global)
frame2d	u_x, u_y, θ_z	$[u_i, v_i, \theta_i, u_j, v_j, \theta_j]$
truss3d	u_x, u_y, u_z	6 (global)
frame3d	all six	12

Internally every member is built once as the full 12-dof 3D frame element, with local dof order $[u_1 \ v_1 \ w_1 \ \theta_{x1} \ \theta_{y1} \ \theta_{z1} \mid u_2 \ v_2 \ w_2 \ \theta_{x2} \ \theta_{y2} \ \theta_{z2}]$, and each analysis type simply selects the rows and columns of its own dofs (`elemSub` in `dofs.js`). One code path serves all five types.

Numbering: free dofs first. If n_f of the n global dofs are unknown, the dofs are numbered $1 \dots n_f$ for the free dofs and $n_f + 1 \dots n$ for the restrained ones (supports, plus dofs restrained automatically, [§9.3](#)), nodes in model order and labels in canonical order. The global vectors are then naturally partitioned,

$$\mathbf{U} = \begin{Bmatrix} \mathbf{U}_f \\ \mathbf{U}_s \end{Bmatrix}, \quad \mathbf{P} = \begin{Bmatrix} \mathbf{P}_f \\ \mathbf{P}_s \end{Bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fs} \\ \mathbf{K}_{sf} & \mathbf{K}_{ss} \end{bmatrix}.$$

(The code uses 0-based indices; the report shows 1-based "code numbers".)

4 Element stiffness matrices

The stiffness matrix $\mathbf{k}'$ of a member relates its local end displacements $\mathbf{u}'$ to the local end forces it needs to hold them: $\mathbf{q}' = \mathbf{k}'\mathbf{u}'$ (for an unloaded member). Column c of $\mathbf{k}'$ is the force vector that produces $u'_c = 1$ and all other $u'_{c'} = 0$.

4.1 Axial and torsional bars

Linear displacement field, $\varepsilon = du/dx = (u_2 - u_1)/L$, $N = EA\varepsilon$:

$$\mathbf{k}'_{\text{axial}} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{k}'_{\text{torsion}} = \frac{GJ}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

4.2 Beam element, 4×4

The Hermite cubic shape functions for $(v_1, \theta_1, v_2, \theta_2)$ at $s = x/L$ are

$$N_1 = 1 - 3s^2 + 2s^3, \quad N_2 = L(s - 2s^2 + s^3), \quad N_3 = 3s^2 - 2s^3, \quad N_4 = L(s^3 - s^2). \quad (4.1)$$

With $v(x) = \mathbf{N}\mathbf{u}'$, the strain energy $\frac{1}{2} \int EI (v'')^2 dx$ gives $k_{ab} = \int_0^L EI N'_a N'_b dx$:

$$\mathbf{k}'_{\text{beam}} = \frac{EI_z}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}, \quad \mathbf{u}' = [v_1, \theta_1, v_2, \theta_2]^T. \quad (4.2)$$

4.3 2D truss element (global 4×4)

An axial-only bar has the local matrix $(EA/L) [1, -1; -1, 1]$ in the dofs (u'_1, u'_2) . With $c = \cos \phi = (x_j - x_i)/L$, $s = \sin \phi = (y_j - y_i)/L$ the axial displacements are $u'_1 = c u_i + s v_i$ etc., and $\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$ becomes

$$\mathbf{k}_{\text{truss2d}} = \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}, \quad \mathbf{u} = [u_i, v_i, u_j, v_j]^T. \quad (4.3)$$

4.4 2D frame element (6×6)

Superposing (4.1) and (4.2) in the dof order $[u_1, v_1, \theta_1, u_2, v_2, \theta_2]$:

$$\mathbf{k}'_{\text{frame2d}} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix}. \quad (4.4)$$

Rotation to global axes uses

$$\mathbf{T} = \begin{bmatrix} \lambda_2 & \mathbf{0} \\ \mathbf{0} & \lambda_2 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{u}' = \mathbf{T}\mathbf{u}, \quad \mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}. \quad (4.5)$$

(The rotation θ_z is the same in both systems because z is common.) The beam element is the same matrix with the axial rows removed and the axis along X .

4.5 3D truss element (global 6×6)

With the direction cosines $\lambda = (l, m, n) = (\mathbf{x}_j - \mathbf{x}_i)/L$:

$$\mathbf{k}_{\text{truss3d}} = \frac{EA}{L} \begin{bmatrix} \lambda\lambda^T & -\lambda\lambda^T \\ -\lambda\lambda^T & \lambda\lambda^T \end{bmatrix}, \quad \lambda\lambda^T = \begin{bmatrix} l^2 & lm & ln \\ lm & m^2 & mn \\ ln & mn & n^2 \end{bmatrix}. \quad (4.6)$$

The member force is $N = (EA/L) \lambda \cdot (\mathbf{u}_j - \mathbf{u}_i)$, tension positive.

4.6 3D frame element (12×12)

In the dof order $[u_1, v_1, w_1, \theta_{x1}, \theta_{y1}, \theta_{z1}, u_2, v_2, w_2, \theta_{x2}, \theta_{y2}, \theta_{z2}]$ the four deformation modes are uncoupled in local axes. Writing $a = 12EI/L^3$, $b = 6EI/L^2$, $c = 4EI/L$, $e = 2EI/L$, the non-zero entries (upper triangle; the matrix is symmetric) are

- axial: $k_{1,1} = k_{7,7} = EA/L$, $k_{1,7} = -EA/L$;
- torsion: $k_{4,4} = k_{10,10} = GJ/L$, $k_{4,10} = -GJ/L$;
- bending about z ($v, \theta_z; I_z$): $k_{2,2} = k_{8,8} = \frac{12EI_z}{L^3}$, $k_{2,6} = k_{2,12} = \frac{6EI_z}{L^2}$, $k_{2,8} = -\frac{12EI_z}{L^3}$, $k_{6,6} = k_{12,12} = \frac{4EI_z}{L}$, $k_{6,8} = k_{8,12} = -\frac{6EI_z}{L^2}$, $k_{6,12} = \frac{2EI_z}{L}$;
- bending about y ($w, \theta_y; I_y$): the same pattern with I_y , but because $\theta_y = -dw/dx$ the w - θ coupling terms change sign: $k_{3,3} = k_{9,9} = \frac{12EI_y}{L^3}$, $k_{3,5} = k_{3,11} = -\frac{6EI_y}{L^2}$, $k_{3,9} = -\frac{12EI_y}{L^3}$, $k_{5,5} = k_{11,11} = \frac{4EI_y}{L}$, $k_{5,9} = k_{9,11} = +\frac{6EI_y}{L^2}$, $k_{5,11} = \frac{2EI_y}{L}$.

(indices are 1-based here.) The transformation is $\mathbf{T} = \text{diag}(\boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda})$ where the **rows** of the 3×3 matrix $\boldsymbol{\lambda}$ are the local unit vectors,

$$\boldsymbol{\lambda} = \begin{bmatrix} \mathbf{e}_x^T \\ \mathbf{e}_y^T \\ \mathbf{e}_z^T \end{bmatrix}, \quad \mathbf{v}' = \boldsymbol{\lambda} \mathbf{v} \text{ (translations and rotations alike),} \quad \mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}. \quad (4.7)$$

The axes $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ follow the local-axis rule of §2, including the vertical-member and roll rules.

5 Assembly

Let member m join nodes i and j and let its **code number** vector $\mathbf{c}_m =$

$[g(i, 1), \dots, g(i, n_d), g(j, 1), \dots, g(j, n_d)]$ list the global dof index of each element dof (this is the "code" column of the report). The global stiffness matrix and the assembled load vectors are

$$K_{c_m(a), c_m(b)} += k_{ab}^{(m)}, \quad FEF_{c_m(a)} += f_a^{(m)}, \quad P_g = \sum \text{joint loads at dof } g \quad (5.1)$$

(spring stiffnesses are added to the diagonal K_{gg} of the dof they act on, §9.1). Because the free dofs are numbered first, $\mathbf{K}_{ff}$, $\mathbf{K}_{fs}$, $\mathbf{K}_{sf}$, $\mathbf{K}_{ss}$ are simply the leading/trailing blocks of $\mathbf{K}$.

The solver stores $\mathbf{K}_{ff}$ in **skyline** (profile) form after a reverse Cuthill–McKee reordering, and factorises it once as $\mathbf{LDL}^T$. Every load case and every combination is then one additional right-hand side, so the stiffness matrix is never re-factored.

6 Partitioned equations, displacements and reactions

Equilibrium of the whole structure, with $\mathbf{P}$ the joint loads and $\mathbf{FEF}$ the assembled fixed-end forces of the member loads (§7), reads $\mathbf{KU} + \mathbf{FEF} = \mathbf{P} + \mathbf{R}$ with $\mathbf{R}$ the support reactions (non-zero only on restrained dofs). Partitioning:

$$\mathbf{K}_{ff}\mathbf{U}_f + \mathbf{K}_{fs}\mathbf{U}_s + \mathbf{FEF}_f = \mathbf{P}_f, \quad \mathbf{K}_{sf}\mathbf{U}_f + \mathbf{K}_{ss}\mathbf{U}_s + \mathbf{FEF}_s = \mathbf{P}_s + \mathbf{R}.$$

Hence

$$\boxed{\mathbf{K}_{ff}\mathbf{U}_f = \mathbf{P}_f - \mathbf{FEF}_f - \mathbf{K}_{fs}\mathbf{U}_s} \quad (6.1)$$

$$\boxed{\mathbf{R} = \mathbf{K}_{sf}\mathbf{U}_f + \mathbf{K}_{ss}\mathbf{U}_s + \mathbf{FEF}_s - \mathbf{P}_s} \quad (6.2)$$

$\mathbf{U}_s$ holds the prescribed support displacements (zero unless a settlement is specified, §10). Reactions are the forces the supports exert on the structure. For a restrained node with an elastic spring on a *free* dof the "reaction" is $-k_s u$.

A structure is **stable** when $\mathbf{K}_{ff}$ is positive definite. A zero or negative pivot during factorisation signals a mechanism (§13).

7 Fixed-end forces of member loads

7.1 Work-equivalent loads

Consider a member whose two ends are **fully fixed** ($\mathbf{u}' = \mathbf{0}$) under a distributed transverse load $q(x)$ (positive in $+y$). The fixed-end force vector $\mathbf{q}_F$ is the set of end forces needed to hold the ends — they are the member-end reactions. By the principle of virtual work with the same Hermite displacement field as the stiffness matrix,

$$\mathbf{q}_F = - \int_0^L \mathbf{N}^T q(x) dx - \sum_k \mathbf{N}^T(a_k) P_k - \sum_k \mathbf{N}'^T(a_k) M_k \quad (7.1)$$

for distributed loads, point forces P_k at $x = a_k$ (positive $+y$) and concentrated couples M_k (positive counter-clockwise). The sign is negative because $\mathbf{q}_F$ are the forces *on the member by the fixed ends*: a downward load ($q < 0$) gives upward end forces. In the structure the member loads are *reversed* into joint loads: a member with end forces $\mathbf{q}_F$ in the fixed state is equivalent to the loading $-\mathbf{q}_F$ on the joints, which is why (6.1) contains $-\mathbf{F}\mathbf{E}\mathbf{F}_f$.

Because (4.1) are the exact solution of the homogeneous beam equation, (7.1) is **exact** for Euler–Bernoulli members, whatever the load. The code integrates distributed loads with a 3-point Gauss rule (exact for the quartic integrand "cubic $\times$ linear"), evaluates $N_a(a)$ for point loads and $N'_a(a)$ for couples, and handles axial and torsional loads with linear shape functions.

7.2 Closed forms (downward loads, magnitudes P, w ; a from node $i, b = L - a$)

Writing $\mathbf{q}_F = [V_i, M_i, V_j, M_j]$ (beam dofs; V up, M counter-clockwise):

LOAD	V_i	M_i	V_j	M_j
point load P at a	$\frac{Pb^2(3a+b)}{L^3}$	$\frac{Pab^2}{L^2}$	$\frac{Pa^2(a+3b)}{L^3}$	$-\frac{Pa^2b}{L^2}$
UDL w over L	$\frac{wL}{2}$	$\frac{wL^2}{12}$	$\frac{wL}{2}$	$-\frac{wL^2}{12}$
triangular, zero at i, w at j	$\frac{3wL}{20}$	$\frac{wL^2}{30}$	$\frac{7wL}{20}$	$-\frac{wL^2}{20}$
triangular, w at i , zero at j	$\frac{7wL}{20}$	$\frac{wL^2}{20}$	$\frac{3wL}{20}$	$-\frac{wL^2}{30}$
couple M (CCW) at a	$\frac{6abM}{L^3}$	$\frac{b(2a-b)M}{L^2}$	$-\frac{6abM}{L^3}$	$-\frac{a(a-2b)M}{L^2}$

Derivation of the point-load row. $\mathbf{N}(a)$ from (4.1) with $s = a/L$: $N_1 = 1 - 3s^2 + 2s^3 = b^2(3a+b)/L^3$, $N_2 = Ls(1-s)^2 = ab^2/L^2$, $N_3 = s^2(3-2s) = a^2(a+3b)/L^3$, $N_4 = -Ls^2(1-s) = -a^2b/L^2$. Then $\mathbf{q}_F = -\mathbf{N}^T(a)(-P) = P\mathbf{N}^T(a)$ — the classic fixed-end values. The couple row follows from $N'_1 = -6ab/L^3$, $N'_2 = b(b-2a)/L^2$, $N'_3 = 6ab/L^3$, $N'_4 = a(a-2b)/L^2$ and $\mathbf{q}_F = -M\mathbf{N}'^T(a)$. For a UDL, $\int_0^L N_1 dx = L/2$, $\int_0^L N_2 dx = L^2/12$, giving the table entries. Equilibrium of the member is a useful check on every row, e.g. for the point load: $V_i + V_j = P$ and $M_i + M_j + V_jL - Pa = 0$.

Loads that are not applied along local axes are first resolved into local components q_x, q_y, q_z (global loads with $\hat{Y}$ etc.; "projected" loads are per unit projected length, so the resolved intensity is $w \cdot L_x/L$ for a load in Y and a member with horizontal extent L_x). Partial and trapezoidal loads are integrated over their actual extent $[x_1, x_2]$.

7.3 Moment loads

A concentrated couple M about local z at $x = a$ gives the last row of the table. In 3D, a couple about local y couples to (w, θ_y) with the sign of $\theta_y = -dw/dx$ taken into account; a couple about x is carried by the torsion dofs with linear shape functions.

7.4 Axial loads, self-weight

An axial distributed load q_x gives $\mathbf{q}_F = -\int \mathbf{N}_a^T q_x dx$ with the linear functions $N = [1 - s, s]$; a uniform q_x gives $-q_x L/2$ at each end. **Self-weight** is the load $w = \rho g A$ per unit length acting in global $-Y$ over the full member length; its total is $\rho g AL$ whatever the inclination.

7.5 Temperature changes and lack of fit

Initial strains are treated like loads: with both ends fixed, the stress-producing part of the strain is the *negative of the free strain*. A member with a free thermal strain $\varepsilon_0 = \alpha \Delta T$ and a free thermal curvature $\kappa_z = -\alpha \Delta T_y / h_y$, held fixed, develops

$$N = -EA \alpha \Delta T, \quad M_z = -EI_z \kappa_z = + \frac{EI_z \alpha \Delta T_y}{h_y} \quad (7.2)$$

(compression for a temperature rise; a sagging moment when the $+y$ face is the hotter one). In terms of the stored fixed-end forces, $N(0) = -q_{u1}$ and $M_z(0) = -q_{\theta z1}$ give

$$\begin{aligned} q_{u1} &= +EA \varepsilon_0, & q_{u2} &= -EA \varepsilon_0, \\ q_{\theta z1} &= +EI_z \kappa_z, & q_{\theta z2} &= -EI_z \kappa_z, \\ q_{\theta y1} &= -EI_y \kappa_y, & q_{\theta y2} &= +EI_y \kappa_y. \end{aligned}$$

A **lack of fit** Δ (member fabricated too long, $\Delta > 0$) is a free strain $\varepsilon_0 = \Delta/L$, so a member forced between fixed points carries

$$N = -\frac{EA \Delta}{L} \quad (\text{compression for a member that is too long}). \quad (7.3)$$

In an unrestrained member a thermal strain produces displacement but no force; the displacement functions of §12 include ε_0 and κ so that deflections are exact.

8 Member end releases (static condensation)

Let the dofs of an element be split into retained (r) and released (c , "condensed") ones. A released dof has zero end force, so the corresponding row of $\mathbf{q} = \mathbf{k}\mathbf{u} + \mathbf{q}_F$ reads $0 = \mathbf{k}_{cr}\mathbf{u}_r + \mathbf{k}_{cc}\mathbf{u}_c + \mathbf{f}_c$ and

$$\mathbf{u}_c = -\mathbf{k}_{cc}^{-1}(\mathbf{k}_{cr}\mathbf{u}_r + \mathbf{f}_c). \quad (8.1)$$

Substituting back into the retained rows gives the condensed stiffness and fixed-end forces,

$$\boxed{\mathbf{k}^* = \mathbf{k}_{rr} - \mathbf{k}_{rc}\mathbf{k}_{cc}^{-1}\mathbf{k}_{cr}, \quad \mathbf{f}^* = \mathbf{f}_r - \mathbf{k}_{rc}\mathbf{k}_{cc}^{-1}\mathbf{f}_c.} \quad (8.2)$$

The condensed matrix keeps its size; rows/columns of released dofs become zero. The code performs one dof at a time (Gaussian elimination), which handles any combination of releases; $\mathbf{u}_c$ is recovered with (8.1) for the diagrams. Example — hinge at node j of a beam ($c = \theta_2$):

$$\mathbf{k}^* = \frac{EI}{L^3} \begin{bmatrix} 3 & 3L & -3 & 0 \\ 3L & 3L^2 & -3L & 0 \\ -3 & -3L & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{UDL: } \mathbf{f}^* = \left[\frac{3wL}{8}, \frac{wL^2}{8}, \frac{5wL}{8}, 0 \right].$$

A **truss (axial-only) member** is the same operation with all bending and torsion dofs released: its matrix is $(EA/L)[1, -1; -1, 1]$, and transverse member loads are condensed to simply-supported end reactions applied at its end nodes (a stiffness of arbitrary positive value is used for the condensation, which does not influence the result). A torsion release at both ends has no stiffness at all; such a dof is simply left without stiffness.

9 Springs and inclined supports

9.1 Elastic springs

A spring of stiffness k_s acting on a free dof g adds k_s to K_{gg} ; it is an extra stiffness element grounded at the node. Its reaction is $-k_s U_g$. Springs may carry any of the six dofs (N/m or N·m/rad).

9.2 Inclined supports and the nodal rotation Γ

A roller on an inclined plane restrains only the displacement normal to the plane. Such a node uses a **support frame** rotated by ϑ (`angle`) about Z ; its dofs are $\mathbf{u}_s = \Gamma \mathbf{u}$ with

$$\Gamma = \begin{bmatrix} \cos \vartheta & \sin \vartheta & 0 \\ -\sin \vartheta & \cos \vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (dofs } u_x, u_y, \theta_z), \quad \mathbf{u} = \Gamma^T \mathbf{u}_s.$$

The support-frame x axis is $(\cos \vartheta, \sin \vartheta)$, the y axis $(-\sin \vartheta, \cos \vartheta)$. Restraints are applied in the support frame: " u_y restrained" gives a reaction **along the support-frame y axis**, i.e. normal to the plane. Every member end at such a node uses the modified transformation $\tilde{\mathbf{T}} = \mathbf{T} \Gamma^T$ so that $\mathbf{u}' = \mathbf{T} \Gamma^T \mathbf{u}_s$ and $\mathbf{k} = \tilde{\mathbf{T}}^T \mathbf{k}' \tilde{\mathbf{T}}$. The solved $\mathbf{U}_f, \mathbf{R}$ are in the support frame at those nodes and are rotated back ($\mathbf{u} = \Gamma^T \mathbf{u}_s$) for reporting; `reactionsLocal` keeps the support-frame components. Inclined supports exist only in the 2D truss and frame analyses.

9.3 Auto-restrained dofs

A rotational dof that has no stiffness — e.g. a node where only axial-only members or fully hinged members meet — would make $\mathbf{K}_{ff}$ singular although it has no physical meaning. Such dofs (and all dofs of nodes not connected to any member) are **restrained automatically** and reported as information (`auto-restrained`). Translational mechanisms are never hidden this way ([§13](#)).

10 Prescribed displacements (settlement)

A support displacement $\bar{\mathbf{U}}_s \neq \mathbf{0}$ (settlement, imposed rotation) enters through $\mathbf{U}_s$ in (6.1) and (6.2). A settlement Δ of one support of a *determinate* structure causes rigid-body motion and no member forces, while in an *indeterminate* structure it induces forces: for a fixed–fixed beam whose right end sinks by Δ , $q_{v1} = 12EI\Delta/L^3$, $q_{\theta1} = q_{\theta2} = 6EI\Delta/L^2$, i.e. moments $6EI\Delta/L^2$ at each end of opposite sense (hogging at the left end, sagging at the right end).

11 Member end forces

After solving, each member's local end displacements are $\mathbf{u}' = \mathbf{T} \tilde{\mathbf{u}}$ (the member's rows of $\mathbf{U}$, in the support frame where applicable) and

$$\boxed{\mathbf{q}' = \mathbf{k}' \mathbf{u}' + \mathbf{q}_F} \quad (11.1)$$

with $\mathbf{k}'$ and $\mathbf{q}_F$ already condensed for any releases. Released end displacements are recovered with (8.1). The member end forces in global axes are $\mathbf{q} = \mathbf{T}^T \mathbf{q}'$.

12 Exact internal-force and deflection functions

The end forces and the member loads are all that is needed to write equilibrium of the free body from 0 to x (with all loads between 0 and x included):

$$\begin{aligned}
 N(x) &= -q_{u1} - \int_0^x q_x d\xi - \sum P_x, \\
 V_y(x) &= q_{v1} + \int_0^x q_y d\xi + \sum P_y, \\
 V_z(x) &= -q_{w1} - \int_0^x q_z d\xi - \sum P_z, \\
 T(x) &= -q_{\theta x1} - \sum T_x, \\
 M_z(x) &= -q_{\theta z1} + \int_0^x V_y d\xi - \sum M_{z,\text{applied}}, \\
 M_y(x) &= -q_{\theta y1} + \int_0^x V_z d\xi - \sum M_{y,\text{applied}},
 \end{aligned} \tag{12.1}$$

with the sums over loads at positions $\leq x$. By construction $V_y = dM_z/dx$ and $V_z = dM_y/dx$. Within each interval between *breakpoints* (member ends, point loads and couples, start/end of distributed loads) every function is an **exact polynomial** in $s = x - x_0$: N and V are one degree above the load, M two. Point forces and couples produce jumps in V and M that are applied at the breakpoints; a distributed trapezoid gives a quadratic V and a cubic M . The segments are stored as coefficient arrays, so extremes are found by root-finding of the derivative, not by sampling.

Deflections follow by integrating the curvature and strain, starting from the member's own end displacements $\mathbf{u}'$ at $x = 0$:

$$\begin{aligned}
 d'_x &= \frac{N}{EA} + \varepsilon_0, & r'_z &= \frac{M_z}{EI_z} + \kappa_z, & d'_y &= r_z, \\
 r'_x &= \frac{T}{GJ}, & r'_y &= \frac{M_y}{EI_y} - \kappa_y, & d'_z &= -r_y.
 \end{aligned} \tag{12.2}$$

For a simply supported beam under a UDL this yields $d_y(L/2) = -5wL^4/(384EI)$; for a cantilever with tip load P (downward) $d_y(L) = -PL^3/(3EI)$, in agreement with the closed forms. **Points of interest** — the extreme values of every function, the zeros of V (location of the extreme moment) and the largest deflection — are found as the roots of the derivative polynomial in each segment, plus the breakpoints themselves.

For a truss member carrying transverse loads, the diagrams are those of a simply supported member under the same load (end reactions applied to the joints).

13 Self-checks

Three checks always run; failures become warnings or errors.

1. **Global equilibrium.** The resultant of the applied loads (joint loads plus the resultants of all member loads, including self-weight) and the reactions, about the global origin, $\sum \mathbf{F}_{\text{applied}} + \sum \mathbf{R} \approx \mathbf{0}$ and $\sum \mathbf{M}_{\text{applied}} + \sum \mathbf{M}_R \approx \mathbf{0}$, with a tolerance relative to the magnitude of the loads (10^{-6}).
2. **End compatibility.** The internal forces of §12 evaluated at $x = L$ must equal the j -end forces: $N(L) = q_{u2}$, $V_y(L) = -q_{v2}$, $M_z(L) = +q_{\theta z2}$, and the integrated deflection at $x = L$ must equal the j -end displacement. This ties the displacement functions to the end forces and verifies the sign conventions end to end.
3. **Instability.** During factorisation of $\mathbf{K}_{ff}$ a zero or negative pivot (relative tolerance 10^{-11}) means that a dof has no stiffness. The solver reports "the structure is unstable (a mechanism)", naming the **node** and the **dof** (and flags inclined support frames), instead of returning garbage.

Further model checks (unknown nodes or sections, zero-length members, dofs a load cannot act on) are reported as errors or warnings by the model validation.

14 Load combinations and envelopes

The structure is linear, so the response to a combination $\sum_k \gamma_k LC_k$ is the factored sum of the case responses. It is computed by applying the same factors to the joint loads, fixed-end forces and prescribed displacements, and solving one more right-hand side with the existing factorisation; displacements, reactions, end forces and the piecewise-polynomial diagrams therefore equal $\sum_k \gamma_k (\text{case result})$, to round-off.

The **envelope** is computed over the combinations (or over the load cases when no combination is defined): at every station x of every member the maximum and minimum of N, V, T, M, d over all combinations, and for every support the maximum and minimum reaction component. Hence at every station the envelope maximum is at least, and the minimum at most, the value of every combination.

15 Limitations and references

Not modelled: geometric non-linearity and second-order effects, buckling, shear deformation (Timoshenko beams), warping torsion and thin-walled sections, tapered members, rigid end zones and joint offsets, non-linear or tension-only/compression-only elements, material non-linearity, dynamic and seismic response, and soil–structure interaction other than linear springs. Results are only as good as the idealisation: the user remains responsible for code checks and design.

References

1. A. Kassimali, *Matrix Analysis of Structures*, 2nd ed., Cengage Learning. (Notation for the stiffness method, member code numbers, partitioning, fixed-end forces, releases, inclined supports.)
2. W. McGuire, R. H. Gallagher, R. D. Ziemian, *Matrix Structural Analysis*, 2nd ed., Wiley. (3D frames, local axes and roll angle, static condensation, work-equivalent loads.)
3. R. C. Hibbeler, *Structural Analysis*, 10th ed., Pearson. (Closed-form fixed-end moments and deflection formulas used as benchmarks.)
4. J. S. Przemieniecki, *Theory of Matrix Structural Analysis*, McGraw-Hill. (Element matrices.)

The numerical results of the JavaScript solver are verified in `tests/solver.test.js` (closed-form benchmarks, derived by hand in the comments) and `tests/crosscheck.test.js` (agreement with the independent Python implementation `python/stiffstruct.py` on 49 models).